Umemura quadric fibrations and maximal subgroups of Cr_n.

everything proj./$\mathbb{C}$.

§1. Algebraic subgroups.

Goal: Study (in some sense) the group
$Bir(X) := \{\varphi : X \dashrightarrow X \mid \varphi : \text{bir'l}\}$

In general it's an enormously complicated group: e.g. $X = \mathbb{P}^2$, Blanc-Furter show that there is no structure of an alg. group on $Bir(X)$, or even an ind-group structure (compatible with the notion "family of morphisms").

Def: An alg. subgroup G of $Bir(X)$ is an alg. group acting rat'l. on X.
- G is called a conn. alg. subgroup (c.a.s) if it is connected as an alg. group
- We say that G is a maximal c.a.s (m.c.a.s) if:
  $$\varphi G \varphi^{-1} \leq H \Rightarrow \varphi G \varphi^{-1} = H$$
  where $\varphi \in Bir(X)$, H: c.a.s.

§2. Known results.

Remark: The simpler X, the more complicated the group $Bir(X)$.
For reference, X: variety of general type, then
$Bir(X) = Aut(X)$: finite group.

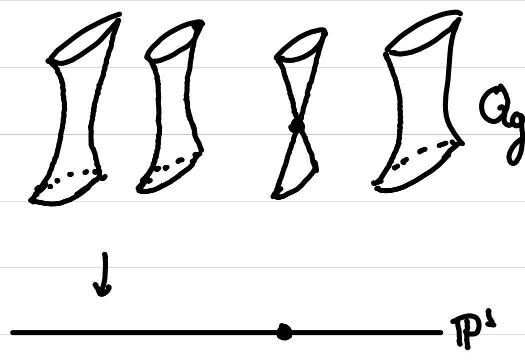
- $X = \mathbb{P}^2$, Enriques 1893:
  G: m.c.a.s $Bir(\mathbb{P}^2) \Leftrightarrow G = Aut^\circ(S)$, $S = \mathbb{P}^2$ or $\mathbb{F}_n$, $n \neq 1$.

- $X = C \times \mathbb{P}^1$ $g(C) \geq 1$, Fong '21
  Full classification of m.c.a.s
  e.g. $g(C) \geq 2$
  G: m.c.a.s. $\Leftrightarrow G = Aut^\circ(C \times \mathbb{P}^1)$

Notice the pattern: $G = Aut^\circ(Z)$, Z: Mfs.

- $X = \mathbb{P}^3$, Umemura 80's, Blanc-Fanelli-Terpereau '21.
  Full classification
  Only one infinite family of m.c.a.s.

Def: $\mathbb{P}(\underbrace{\mathcal{O}_{\mathbb{P}^1}^{\oplus n} \oplus \mathcal{O}_{\mathbb{P}^1}(-a)}_{\mathcal{E}_a}) = (\mathbb{A}^{n+1} \setminus \{0\}) \times (\mathbb{A}^2 \setminus \{0\}) / \mathbb{G}_m^2$
$(\lambda, \mu) \cdot (x_0, \dots, x_n)(t_0, t_1) = (\lambda x_0, \dots, \lambda x_{n-1}, \lambda \mu^{-a} x_n)(\mu t_0, \mu t_1)$
Let g be a polynomial of deg $2a$. We define the Umemura quadric fibration
$$Q_g := \{x_1^2 - x_0 x_2 + x_3^2 + \dots + x_{n-1}^2 + g(t_0, t_1) x_n^2 = 0\} \subseteq \mathbb{P}(\mathcal{E}_a)$$



Facts
- singular fibers = fibers over roots of g
- if p is a multiple root then vertex of fibers $\leq Sing(Q_g)$
- $Q_g \to \mathbb{P}^1$: Mfs and Q_g: rat'l.

Thm (Floris, -): Write $g = f^2 h$, h: square-free.
Then $Aut^\circ(Q_g) \leq Aut^\circ(Q_h)$.
Let h, h' be square-free polynomials. Then
(1) $Aut^\circ(Q_h)$: m.c.a.s of Cr_n iff h: constant or h has at least 4 roots.
(2) $Aut^\circ(Q_h)$, $Aut^\circ(Q_{h'})$ are conjugated iff $h(t_0, t_1) = h'(\alpha(t_0, t_1))$, $\alpha \in PGL_2$.

§3. Main machinery

- How to study c.a.s.

<u>Thm</u> (Weil regularization +): A c.a.s. G of $\mathrm{Bir}(X)$ is regularizable on a smooth proj. variety, i.e. there exist:

- Y: smooth proj.
- $G \curvearrowright Y$
- $X \dashrightarrow Y$: G-equivariant.

<u>Prop</u>: Let G: conn alg. group acting on Y. Then any MMP on Y is G-equivariant.

<u>Proof</u>: $\left.\begin{array}{r}\text{The } K_Y\text{-negative part of } \overline{NE}(Y) \text{ is discrete}\\ G\text{: connected}\end{array}\right\} \Rightarrow$

$\Rightarrow G \curvearrowright \overline{NE}(Y)_{K_Y<0}$: trivially

$\Rightarrow$ Any extr. contraction is G-equiv.

- Divisorial contractions ✓
- Flips $\begin{tikzcd} Y \arrow[rr, dotted] \arrow[dr] & & Y^+ \arrow[dl] \\ & W & \end{tikzcd}$ $\left.\begin{array}{l}\text{- } G \curvearrowright W\\ \text{- } G\text{: preserves sections of } nK_Y,\ n \gg 0\end{array}\right\} \Rightarrow G \curvearrowright Y^+$ □

$G \overset{rat}{\curvearrowright} X \rightsquigarrow G \curvearrowright Y \rightsquigarrow G \curvearrowright Z$, Z: terminal Mfs birt'l X.

- How to check maximality of a c.a.s

<u>Lemma</u>: G: c.a.s of $\mathrm{Bir}(X)$, then
G: maximal $\Leftrightarrow \forall$ ~~G-equiv. birt'l map~~ G-Sarkisov link $f: \cancel{X} \overset{Z' \text{ Mfs}}{\dashrightarrow} Z$, to a Mfs with a G-action
$G = \mathrm{Aut}^\circ(Z)$

<u>Proof</u>:

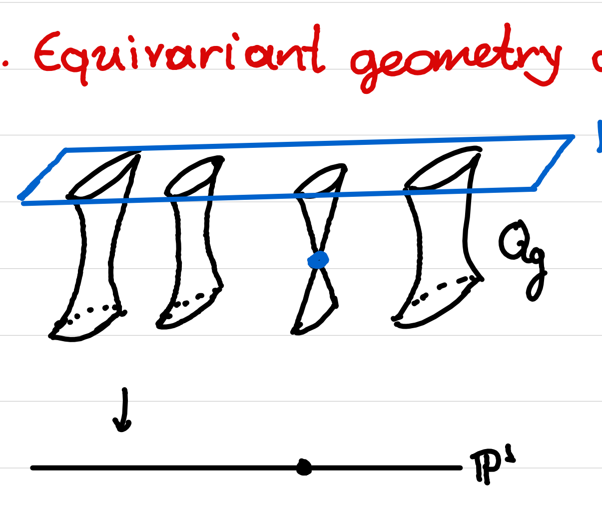
$(\Rightarrow)$ $\left.\begin{array}{l} G \leq \mathrm{Aut}^\circ(Z)\\ \mathrm{Aut}^\circ(Z)\text{: c.a.s}\end{array}\right\} \Rightarrow G = \mathrm{Aut}^\circ(Z)$

$(\Leftarrow)$ $G \leq M$: c.a.s.
$\left.\begin{array}{l} M \overset{rat}{\curvearrowright} X \rightsquigarrow G \leq M \curvearrowright Z \text{ : Mfs}\\ \text{by our assuption}\\ G \leq \mathrm{Aut}^\circ(Z)\end{array}\right\} \Rightarrow G = M.$ □

<u>Thm</u> (Corti, Hacon-McKernan, Floris): A G-equiv. birt'l map between Mfs with a G-action can be decomposed into simpler ones, called G-Sarkisov links, where G is connected.

<u>Remark</u>: All G-Sarkisov links involving $Q_g \to \mathbb{P}^1$ are determined by a G-equiv. divisorial contraction $X \to Q_g \to \mathbb{P}^1$

§4. Equivariant geometry of Q_g.



Lemma:

1. Closed orbits of $\mathrm{Aut}^\circ(Q_g)_{\mathbb{P}^1}$ are:
   - vertices of singular fibers
   - $H_n \cap$ fibers
2. If g has more than 3 distinct roots, then $\mathrm{Aut}(Q_g) = \mathrm{Aut}(Q_g)_{\mathbb{P}^1} \cong SO_n(\mathbb{C})$.
3. Singular fibers are preserved.

Two examples of G-links

- Suppose that $g = t_0^2 \cdot h$, p: vertex of F_{t_0}

$$\begin{tikzcd} & X = \mathrm{Bl}_p Q_g \arrow[dl] \arrow[dr] & \\ Q_g \arrow[d] & & Q_h \arrow[d] \\ \mathbb{P}^1 \arrow[rr, equal] & & \mathbb{P}^1 \end{tikzcd}$$

in particular

$$\mathrm{Aut}^\circ(Q_g) \leq \mathrm{Aut}^\circ(Q_h)$$

in coordinates: $(x_0, \dots, x_n; t_0, t_1) \mapsto (x_0, \dots, x_{n-1}, t_0 x_n; t_0, t_1)$

- Suppose that $g = t_0 t_1$, the H_n: closed orbit

$$\begin{tikzcd} Q_g \arrow[d] \arrow[dr] & \\ \mathbb{P}^1 \arrow[d] & Q^n \subseteq \mathbb{P}^{n+1} \arrow[dl] \\ \mathrm{Spec}(\mathbb{C}) & \end{tikzcd}$$

in coordinates: $(x_0, \dots, x_n; t_0, t_1) \mapsto (x_0 : \dots : x_{n-1} : x_n t_0 : x_n t_1)$
$\{x_n = 0\} \longmapsto \{y_n = y_{n+1} = 0\} = \Pi$

$$\mathrm{Aut}^\circ(Q_g) = \mathrm{Aut}^\circ(Q^n; \Pi) \lneq \mathrm{Aut}^\circ(Q^n)$$

in particular $\mathrm{Aut}^\circ(Q_g)$: not m.c.a.s.

<u>Prop</u>: Let $f: E \subseteq X \to \Gamma \subseteq Y$ be a G-equiv div. contraction, where X: terminal and Γ is an orbit of codim 2.
Then $X = \mathrm{Bl}_\Gamma Y$.

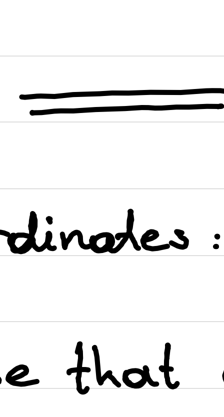
<u>Prop</u>: Suppose $g = t^k \cdot h$, $k \geq 1$, i.e. fiber F_t / t is singular and let $f: E \subseteq X \to p \in Q_g$ be a G-equiv div. contraction, p: vertex of F_t
Then f is a $(1, 1, \dots, b)$-weighted blowup.

In this case write $\overline{NE}(X/\mathbb{P}^1) = \mathbb{R}_+[e] \oplus \mathbb{R}_+[\tilde{\ell}]$, $e \subseteq N$
Then:
- $\tilde{\ell}$ spans the divisor $\tilde{F}_t$
- $K_X \cdot \tilde{\ell} < 0 \Leftrightarrow b = 1$ and $k \geq 2$

<u>Idea of Proof</u>: Tower construction of Kawakita

$$\begin{tikzcd} & E_m \subseteq X_m \arrow[dl, "\varphi"'] \arrow[d, dotted] \\ E \subseteq X \arrow[dr] & E_1 \subseteq X_1 \arrow[d] \\ & p \in Q_g \end{tikzcd}$$

- Each blowup $E_i \subseteq X_i \to z_{i-1} \subseteq X_{i-1}$ is centered at $z_{i-1} \subseteq \mathrm{Exc}(X_i \to Q_g)$
- z_i: orbit.
- orbits are always of the form $E_N \cap E_S$ so $\mathrm{Exc}(X_m \to Q_g)$ is a "tree"



- φ: morphism, thus we can assume it's the output of some MMP.
- we analize K_{X_m}-negative rays and the only configuration that works is the $(1, \dots, 1, b)$-blowup. □

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(2) $\mathrm{Aut}^\circ(Q_h), \mathrm{Aut}^\circ(Q_{h'})$ are conjugated iff $h(t_0, t_1) = h'(\alpha(t_0, t_1))$, $\alpha \in PGL_2$.

<u>Proof</u>: The only links are the 2 examples

- If $g = f^2 \cdot h$, h: sq. free with 2 roots

  Then repeatedly applying the 1st link we get
  $$\mathrm{Aut}^\circ(Q_g) \leq \mathrm{Aut}^\circ(Q_h)$$
  applying the 2nd link
  $$\mathrm{Aut}^\circ(Q_h) \lneq \mathrm{Aut}^\circ(Q^n)$$
  i.e. $\mathrm{Aut}^\circ(Q_g)$ not m.c.a.s.

- Otherwise write $g = f^2 \cdot h$, h: sq. free with more than 2 roots.
  From Q_g we can only apply the 1st link or its inverse
  i.e we only have the possibilities
  $Q_g \dashrightarrow Q_{g\ell^2}$ or $Q_g \dashrightarrow Q_{g/\ell^2}$

  All these polynomials are of the form $g' = f' \cdot h$ and thus they have more than 2 roots.
  This means that
  $$\mathrm{Aut}^\circ(Q_{g'}) = SO_n$$ ■